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# Velocity-derivative skewness and two-time velocity correlations of isotropic turbulence as predicted by the LET theory

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The local-energy-transfer (LET) theory was used to calculate freely decaying turbulence for arbitrary initial conditions over a range of microscale-based Reynolds numbers $0.5 \leqslant R_\lambda(t_f) \leqslant 1009$, where $t_f$ is the final time of computation. The predicted skewness factor $S(R_\lambda)$ agreed closely with the results of numerical simulations at low-to-moderate Reynolds numbers and followed the same general trend at larger values of $R_\lambda$. It was also found that, for $R_\lambda(t_f) \leqslant 5$, the LET calculation was almost indistinguishable from that of the direct-interaction approximation (DIA), with the difference between the two theories tending to zero as $R_\lambda(t_f) \to 0$.

Two-time correlation and propagator (or response) functions were also obtained. Tests of their scaling behaviour suggest that, contrary to general belief, the convective sweeping of the energy-containing range is much less important than the Kolmogorov timescale in determining inertial-range behaviour. This result raises questions about the accepted explanation for the failure of the direct-interaction approximation, thus motivating a discussion about the relevance of random Galilean invariance (RGI). It is argued that, for a properly constructed ensemble of transformations to inertial frames, invariance in every realization necessarily implies RGI. It is suggested that the defects of the direct-interaction approximation can be understood in terms of a failure to renormalize the stirring forces.

## 1. Introduction

In a previous paper (McComb & Shanmugasundaram 1984, hereinafter referred to as I) we presented some numerical calculations of decaying isotropic turbulence, using the LET theory (McComb 1978). Results were obtained for energy, dissipation and energy-transfer spectra at low-to-moderate microscale Reynolds numbers ($15 \leqslant R_\lambda(t_f) \leqslant 40$) and at a single large Reynolds number ($R_\lambda(t_f) = 533$), where $t_f$ is the final time of computation. Associated integral parameters were also calculated, and comparisons were made with experimental results and with other theories. In this paper we present more extensive calculations of the velocity-derivative skewness (or, simply, skewness factor), and of velocity correlations evaluated at two different times.

The LET (local energy transfer) theory is a two-point, two-time Eulerian closure.

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In I we noted that for $R_\lambda(t_f) < 40$ the predictions of the LET theory agree well with experimental results, and with the predictions of the well-known direct-interaction approximation (DIA) (in Eulerian form: Kraichnan 1959). The main difference between the two theories was that LET gave higher values of energy transfer and of evolved skewness factor than DIA.

At $R_\lambda = 533$, the predictions of LET for spectra also agreed well with experiment, and with the two Lagrangian-history forms of DIA, ALHDI and SBALHDI (Herring & Kraichnan 1979). In particular, all three theories yielded the $k^{-\frac{5}{3}}$ law for the inertial range of the energy spectrum at this large value of the Reynolds number. However, the values of the evolved skewness factor were quite different in the three cases, with LET giving $S(t_f) \sim 0.35$, whereas ALHDI and SBALHDI gave $S(t_f) \sim 0.76$ and 0.55 respectively.

In I we suggested that the general trend of experimental results for $S(t_f)$ with $R_\lambda$ favoured the LET value. Yet, we also concluded that the large scatter in the experimental data implied that none of the theoretical values could be said to be incompatible with experiment. In the circumstances, a more detailed investigation of the way in which $S(t_f)$ depends on the Reynolds number $R_\lambda(t_f)$ seemed to be indicated.

In this paper, we present the prediction of the LET theory for the continuous variation of skewness factor $S(t_f)$ with Reynolds number in the range $0.5 \leqslant R_\lambda(t_f) \leqslant 1009$. This is compared with the predictions of other theories and the results from direct numerical simulations. We also hope to shed some light on the relationship between LET and other theories by presenting results for the two-time correlation coefficient. In particular, we investigate the effect of both convective and inertial-range (Kolmogorov) timescaling as a function of Reynolds number.

## 2. Basic equations

The LET theory is a two-point, two-time closure for the (Fourier-transformed) velocity covariance $Q(k;t,t')$ in terms of the renormalized propagator function $H(k;t,t')$. The velocity covariance is related to the familiar energy spectrum by

$$E(k;t) = 4\pi k^2 Q(k;t,t), \tag{2.1}$$

and $H(k;t,t')$ relates the velocity field of mode $k$ to itself at subsequent times.

The above forms are for isotropic turbulence and with this restriction, the LET equations take the form:

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) H(k;t,t') = W(k;t,t') \quad (t' < t), \tag{2.2}$$

and

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) Q(k;t,t') = P(k;t,t'), \tag{2.3}$$

where $\nu$ is the kinematic viscosity of the fluid. The inertial transfer terms $W$ and $P$ are given by

$$W(k;t,t') = \int \mathrm{d}^3\boldsymbol{j} L_{kj\mu} \left[ \int_0^{t'} \mathrm{d}s\, H(k;t',s)\, Q(j;t,s)\, Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s) \right.$$
$$\left. - \int_0^t \mathrm{d}s\, H(k;s,t')\, H(j;t,s)\, Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s) \right] \tag{2.4}$$

and

$$P(k;t,t') = \int \mathrm{d}^3\boldsymbol{j}\, L_{kj\mu}\left[\int_0^{t'} \mathrm{d}s\, H(k;t',s)\, Q(j;t,s)\, Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s) - \int_0^{t} \mathrm{d}s\, Q(k;s,t')\, H(j;t,s)\, Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s)\right], \quad (2.5)$$

where

$$L_{kj\mu} = \frac{[\mu(k^2+j^2)-kj(1+2\mu^2)]\,(1-\mu^2)\,kj}{k^2+j^2-2\mu kj} \quad (2.6)$$

and $\mu$ is the cosine of the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{j}$.

It should be noted that (2.3) and (2.5) are identical (notational differences aside) to the corresponding DIA equations, whereas (2.2) and (2.4) differ from their DIA counterparts by the presence of the first term (i.e. the one containing $Q^{-1}$) on the right-hand side of (2.4) for $W(k;t,t')$. It should also be noted that the first term on the right-hand side of (2.4) differs from the version given in I, insofar as $Q(k;t',t')$ replaces $Q(k;s,s)$ in the denominator. The form in I was an incorrect generalization of the original derivation for stationary turbulence (McComb 1978). A more recent, general derivation of the time-dependent LET theory will be treated elsewhere, but here we note that all calculations with the original equation have been repeated with the new form, and that the numerical differences were in fact negligible.

If we introduce the energy transfer spectrum $T(k,t)$ by the relation

$$T(k,t) = 8\pi k^2 P(k;t,t), \quad (2.7)$$

then the skewness factor is given (Batchelor 1971) by

$$S(t) = \frac{2}{35}\left[\frac{\lambda(t)}{u(t)}\right]^3 \int_0^\infty k^2 T(k,t)\,\mathrm{d}k, \quad (2.8)$$

where $u(t)$ is the r.m.s. value of any velocity component, $\lambda(t)$ is the Taylor microscale (see I for the defining relationships used) and the microscale Reynolds number is defined as

$$R_\lambda(t) = \frac{\lambda(t)\,u(t)}{\nu}. \quad (2.9)$$

Finally, the correlation coefficient is introduced through the relationship

$$R(k;t,t') = \frac{Q(k;t,t')}{[Q(k;t,t)\,Q(k;t',t')]^{\frac{1}{2}}}. \quad (2.10)$$

## 3. Results and discussion

Equations (2.2)–(2.5) were integrated forward in time from arbitrarily prescribed initial spectra. Both wavenumber and time were discretized, and the integrations performed numerically. Full details of the numerical methods were given in I, along with demonstrations of self-preservation and similarity (i.e. independence of initial conditions), and discussions of the errors involved in the computation. As in I, equivalent calculations of the DIA equations were also carried out for the same range of Reynolds numbers. This allowed us to check our numerical procedures by comparing our results for DIA with those of other workers (in this case Herring & Kerr 1982).

The trial spectra may be written in the form

$$E(k,0) = c_1 k^{c_2} \exp(-c_3 k^{c_4}) \quad (3.1)$$

| Run no. | Basic spectral shape | Values of $c$ in the initial spectrum shape: $E(k,0) = c_1 k^{c_2} \exp(-c_3 k^{c_4})$ | | | | $k_{bot}$ | $k_{top}$ | $\nu$ | $t_t$ | $R_\lambda(t_t)$ | $S(t_t)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $c_1$ | $c_2$ | $c_3$ | $c_4$ | | | | | | |
| 1 | I | $0.524169 \times 10^{-2}$ | 4 | $0.883881 \times 10^{-1}$ | 2 | 1.8340 | 29.3440 | 0.01189 | 0.8409 | 14.98 | 0.4931 |
| 2 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.01 | 0.6517 | 16.85 | 0.4870 |
| 3 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.015 | 0.7918 | 12.35 | 0.5011 |
| 4 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.25 | 0.3948 | 4.65 | 0.4608 |
| 5 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 0.4585 | 20.7493 | 0.20 | 0.6271 | 1.41 | 0.2078 |
| 6 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 0.4585 | 20.7493 | 0.60 | 0.1750 | 0.47 | 0.0676 |
| 7 | IV | 0.4 | 1 | 0.5 | 1 | 0.2806 | 35.9166 | 0.008 | 1.0001 | 38.61 | 0.4230 |
| 8 | V | 1.0 | $-1.666667$ | 0 | 0 | 0.1114 | 71.8614 | 0.01 | 1.0198 | 276.40 | 0.3699 |
| 9 | Herring's test problem | $E(k,0) = 2\pi k^2 (0.02+k)^{-\frac{11}{3}}$ | | | | 0.01 | 103.2126 | 0.005 | 0.6051 | 1008.78 | 0.3689 |

TABLE 1. Values of evolved $R_\lambda$ and skewness: results from LET theory

and reference should be made to I, figure 1, for the various shapes used. Initial conditions for a run were set by choosing values for the constants $c_1$–$c_4$, along with a value for the fluid kinematic viscosity $\nu$. Values of these constants for each run are given in Table 1, along with the evolved values of $S$ and $R_\lambda$ for LET. Corresponding results for DIA are given in Table 2. We also used a trial spectrum suggested by J. R. Herring (1987, private communication), and details of this are given in both tables.

### 3.1. *The skewness factor*

Our results for $S(t_f)$ as a function of $R_\lambda(t_f)$ are presented in figure 1. The LET prediction is shown as a continuous line, while the 'spot values' for the individual calculations may be found in table 1. Two points in particular should be noted about the present calculations. First, as $R_\lambda(t_f)$ becomes small our LET and DIA values tend to agree closely with each other. Second, the LET values pass through a small, but definite, peak in the neighbourhood of $R_\lambda(t_f) = 10$ before settling down to a constant value as $R_\lambda(t_f)$ becomes large.

In figure 1 we compare our values for the skewness to results obtained by direct numerical simulations (Siggia & Patterson 1978; Clark, Ferziger & Reynolds 1979; Herring & Kerr 1982; and Kerr 1985). In particular, the latter two references, when taken together provide data for the skewness factor over the range $0.46 \leqslant R_\lambda(t_f) \leqslant 82.32$. Evidently, the LET predictions agree quite well with these simulations in the approximate range $3 < R_\lambda(t_f) < 15$, so we shall consider the two cases $R_\lambda(t_f) < 3$ and $R_\lambda(t_f) > 15$ as separate issues.

When we first did our calculations at Reynolds numbers $R_\lambda(t_f) < 3$, we employed the wavenumber range $1.1 < k < 20.8$, and obtained very close agreement with Herring & Kerr (1982), who used the range $2 < k < 32$. For example, typical values of evolved Reynolds number and skewness were $R_\lambda(t_f) = 0.37$ and $S(t_f) = 0.066$. However, in the process of checking the effect of the lower wavenumber cut-off on the energy content of the spectrum, we repeated the above calculation with $0.46 < k < 20.8$ to obtain $R_\lambda(t_f) = 0.48$, $S(t_f) = 0.067$; and with $0.23 < k < 20.8$ to obtain $R_\lambda(t_f) = 0.50$, $S(t_f) = 0.067$. Thus, while the evolved skewness is (as one would expect) insensitive to changes in the lower wavenumber cut-off, the evolved microscale Reynolds number does change to some extent; and this affects the shape of the graph at low $R_\lambda(t_f)$.

Therefore, tentatively, we are inclined to suppose that at least some of the disagreement in this range between LET and the direct numerical simulation may result from a truncation error, due to Herring & Kerr (1982) taking a relatively large value for the lower cut-off wavenumber. This view receives some additional support if one considers the calculations of Herring & Kerr (1982) for DIA and TFM (test-field model: a single time modification of DIA which is compatible with the Kolmogorov spectrum for the inertial range). Both calculations of DIA agree very well over the range $3 < R_\lambda(t_f) < 40$, but for $R_\lambda(t_f) < 3$ our present calculation of skewness using DIA falls off more steeply as $R_\lambda(t_f)$ becomes small.

The choice of initial spectra was not the same for both our investigation and the direct numerical simulation. In terms of (3.1), our results for low-to-moderate Reynolds number were based on $c_2 = c_4 = 1$, with the values of $c_1$ and $c_3$ being varied as appropriate. In the case of Herring & Kerr (1982), initial spectra were based on $c_2 = 4$, $c_4 = 2$. However, as noted above, our calculations agreed with theirs when we took a larger value for the lower cut-off wavenumber.

Turning now to the situation where the Reynolds number exceeds a value of about 15, we see that the direct numerical simulation of Kerr (1985) leads to the skewness

| Run no. | Basic spectral shape | Values of $c$ in the initial spectrum shape: $E(k,0) = c_1 k^{c_2} \exp(-c_3 k^{c_4})$ | | | | $k_{bot}$ | $k_{top}$ | $\nu$ | $t_f$ | $R_\lambda(t_f)$ | $S(t_f)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $c_1$ | $c_2$ | $c_3$ | $c_4$ | | | | | | |
| 1 | I | $0.524169 \times 10^{-2}$ | 4 | $0.883881 \times 10^{-1}$ | 2 | 1.8340 | 29.3440 | 0.01189 | 0.8215 | 16.59 | 0.4257 |
| 2 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.01 | 0.6462 | 17.44 | 0.4183 |
| 3 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.015 | 0.7901 | 12.76 | 0.4118 |
| 4 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 1.0905 | 41.4989 | 0.05 | 0.3948 | 4.69 | 0.3562 |
| 5 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 0.4585 | 20.7493 | 0.20 | 0.6736 | 1.41 | 0.1872 |
| 6 | III | $0.662912 \times 10^{-1}$ | 1 | 0.210224 | 1 | 0.4585 | 20.7493 | 0.60 | 0.1750 | 0.47 | 0.0669 |
| 7 | IV | 0.4 | 1 | 0.5 | 1 | 0.2806 | 35.9166 | 0.008 | 0.9936 | 40.50 | 0.4081 |
| 8 | V | 1.0 | $-1.666667$ | 0 | 0 | 0.1114 | 71.8614 | 0.01 | 1.0194 | 282.59 | 0.3383 |
| 9 | Herring's test problem | $E(k,0) = 2\pi k^2 (0.02 + k)^{-\frac{11}{3}}$ | | | | 0.01 | 103.2126 | 0.005 | 0.6048 | 1039.31 | 0.3439 |
| 10 | VII | 6.29 | $-1.5$ | 0 | 0 | 0.1114 | 114.0728 | 0.008 | 0.2266 | 369.18 | 0.3494 |
| 11 | VII | 10.49 | $-1.5$ | 0 | 0 | 0.1114 | 228.1456 | 0.005 | 0.0405 | 402.21 | 0.3412 |
| 12 | VII | 104.83 | $-1.5$ | 0 | 0 | 0.1114 | 1149.7799 | 0.002 | 0.0035 | 939.56 | 0.3402 |

TABLE 2. Values of evolved $R_\lambda$ and skewness: results from DIA

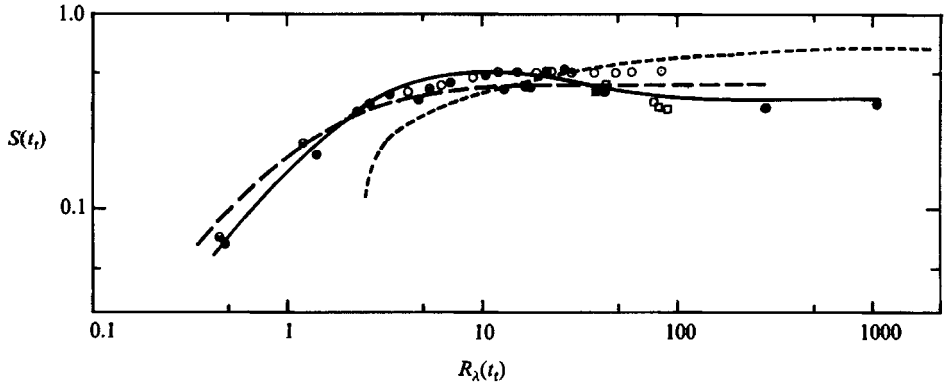


FIGURE 1. Comparison of the evolved skewness factor with numerical simulations and with results from other theories: ——, LET; ●, DIA (present results); — —, DIA (Herring & Kerr 1982); -----, TFM (Herring & Kerr 1982); □, Siggia & Patterson (1978); ■, direct numerical simulation (Clark *et al.* 1979); ⊗, direct numerical simulation (Herring & Kerr 1982); ○, direct numerical simulation (Kerr 1985).

becoming independent of $R_\lambda(t_f)$ and reaching an apparent asymptote of $S \sim 0.51$ for $18.3 < R_\lambda(t_f) < 82.3$. In the same region, the LET value of the skewness declines below Kerr's results and then reaches an asymptote of $S(t_f) = 0.36$ for $146.8 < R_\lambda(t_f) < 1009$.

We also show four points, from the simulation of Siggia & Patterson (1978), which agree more closely with the LET values. Unfortunately, the three points in the neighbourhood of $R_\lambda(t_f) = 80$ may be affected by high-wavenumber truncation errors and are possibly too low. But, the value of $S(t_f) = 0.43$ at $R_\lambda(t_f) = 42$ comes from what these authors call their 'honest' simulation and presumably cannot be faulted in that respect. Some support for this result comes from the simulation by Clark *et al.* (1979), but evidently we need more independent numerical investigations in order to arrive at an agreed curve for skewness against Reynolds number.

The latter remark is underlined by a reference to the array of experimental results compiled by Tavoularis, Bennett & Corrsin (1978). As we have mentioned before, the scatter in the experimental data is such as to rule out any critical comparison of theories. In particular, the very high experimental values of the skewness are normally attributed to non-Gaussian behaviour, such as stratification effects, which certainly would invalidate a comparison with LET (or similar theories).

### 3.2. *Time-correlation functions*

The LET results for the two-time correlations are presented in figures 2–5 for $R_\lambda = 4.7$, 38.6, 834 and 1009 respectively. It should be noted that figure 4 is a special case in that $t_{\rm ref}$ is not equal to $t_f$. We shall return to this point later. In figures 6–8 we give the corresponding results for DIA for $R_\lambda = 4.7$, 40.5 and 1039. In all cases, the same format is used and we shall now explain this, referring to (2.10) for the definition of $R(k;t,t')$.

First, we put $t' = t_{\rm ref}$, where $t_{\rm ref}$ is a fixed reference time, and in most cases, the final time for the calculation. Then, for a given value of wavenumber $k$, we may plot $R(k;t,t_{\rm ref})$ as a function of a single variable $(t_{\rm ref}-t)$, and a fairly complete picture of the function $R(k;t,t_{\rm ref})$ can be built up in this way, by repeating the process for additional values of $k$. Thus, referring to figure 2(*a*), it may be seen that we have plotted curves of $R(k;t,t_{\rm ref})$ versus $(t_{\rm ref}-t)$ for seven values of $k$, in this case spanning

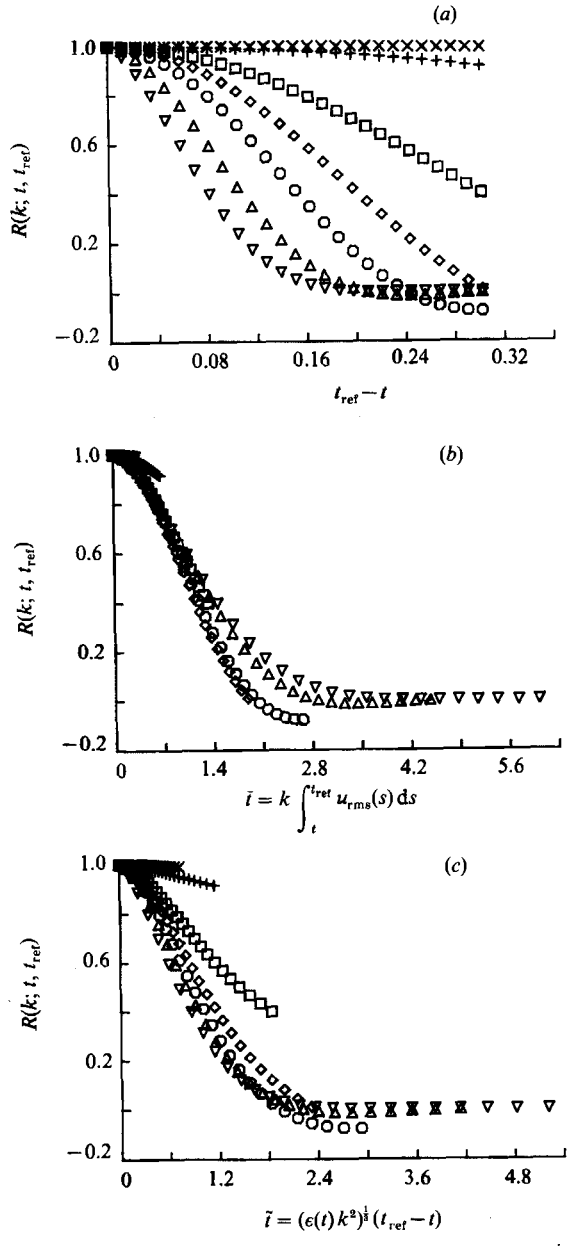


FIGURE 2. LET two-time correlation function for $R_\lambda(t_{ref}) = 4.7$, $t_{ref} = 0.39$, for various wavenumbers: ×, 2.00; +, 4.00; □, 8.00; ◇, 11.31; ○, 16.00; △, 26.91; ▽, 38.05. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

the range $2.00 \leqslant k \leqslant 38.05$. This method of presenting unscaled results has been followed in part (*a*) of each of figures 2–8.

In parts (*b*) and (*c*) of each of these figures, we test the effects of convective scaling and Kolmogorov scaling respectively. That is, in figures 2 (*b*)–8 (*b*) we plot $R(k; t, t_{ref})$ against $\bar{t}$, where

$$\bar{t} = k \int_t^{t_{ref}} u_{rms}(s)\,ds. \tag{3.2}$$

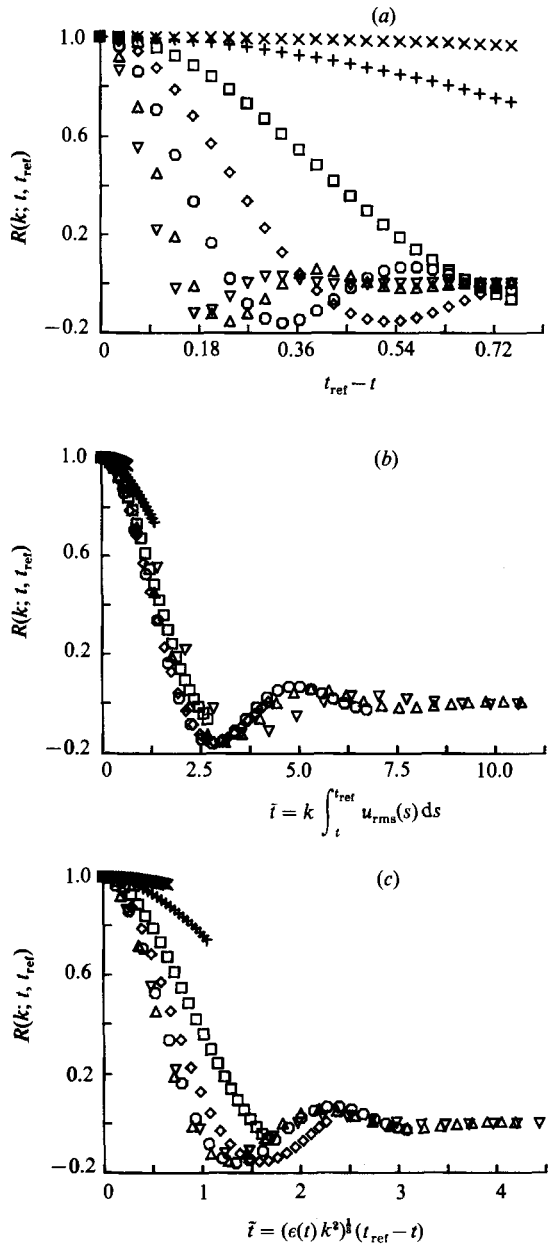


FIGURE 3. LET two-time correlation function for $R_\lambda(t_{ref}) = 38.6$, $t_{ref} = 1.0$, for various wavenumbers: ×, 1.00; +, 2.00; □, 4.00; ◇, 6.35; ○, 10.08; △, 16.00; ▽, 25.40. (*a*) Unscaled results, (*b*) convective scaling. (*c*) inertial scaling.

In using (3.2), we have followed the example of Orszag & Patterson (1972). This takes account of the fact that the turbulence is not stationary, although, in practice, we found very little difference between (3.2) and the usual form $\bar{t} = k u_{rms}(0)\,(t_{ref} - t)$. Hence, in figures 2(*b*)–8(*b*), one is looking for evidence of a relationship of the form

$$R(k; t, t_{ref}) = f(\bar{t}), \tag{3.3}$$

where $f$ is an unknown function.

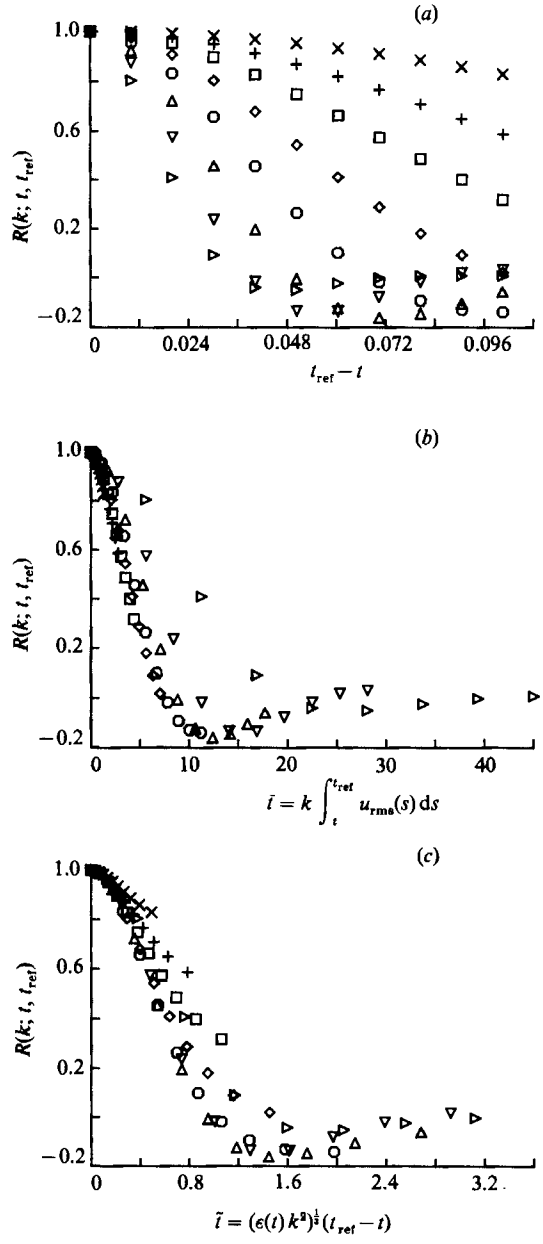


FIGURE 4. LET two-time correlation function for $R_\lambda(t_{ref}) = 834.1$, $t_{ref} = 0.1$, for various wavenumbers: ×, 2.28; +, 4.56; □, 7.24; ◇, 11.49; ○, 18.24; △, 28.96; ▽, 45.97; ▷, 91.95. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

In figures 2(*c*)–8(*c*), $R(k; t, t_{ref})$ is plotted against the scaled variable

$$\tilde{t} = [\epsilon(t)\,k^2]^{\frac{1}{3}}(t_{ref} - t). \tag{3.4}$$

In this case, a good collapse of the data onto one universal curve would imply

$$R(k; t, t_{ref}) = f(\tilde{t}). \tag{3.5}$$

Referring to these figures, an immediate observation is that neither scaling

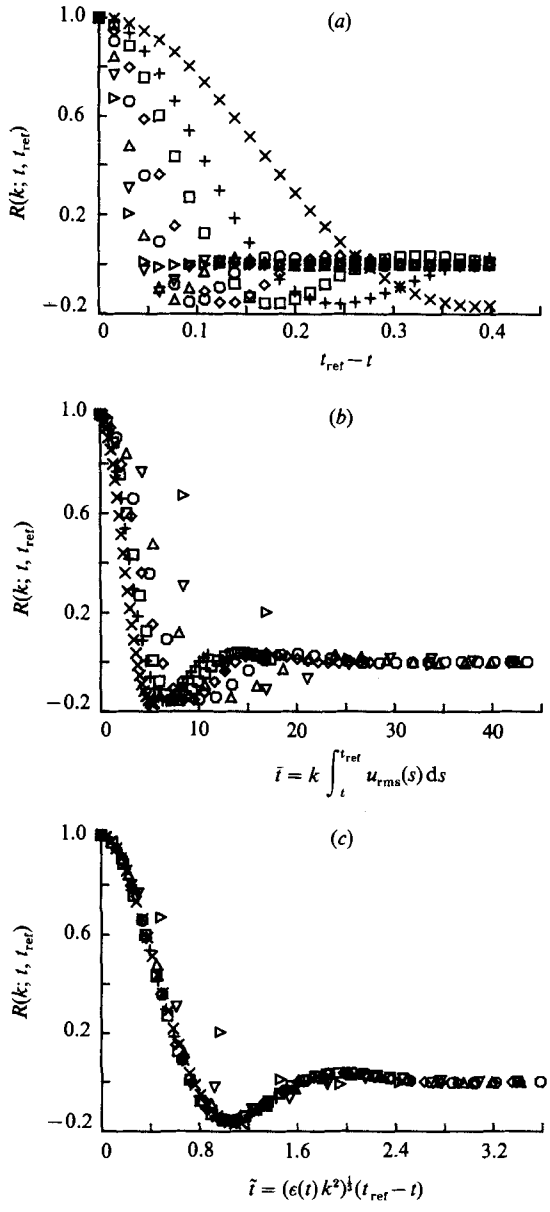


FIGURE 5. LET two-time correlation function for $R_\lambda(t_{ref}) = 1009$, $t_{ref} = 0.6$, for various wavenumbers: ×, 2.28; +, 4.56; □, 7.24; ◇, 11.49; ○, 18.24; △, 28.96; ▽, 45.97; ▷, 91.95. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

transformation is completely successful in collapsing all the data to a single curve. Yet, several plausible conclusions may be drawn from a comparison of the three figures in each case. We list these as follows:

(1) Both methods are partially successful, in that the convective scaling tends to collapse data for the lower wavenumbers, whereas the Kolmogorov scaling is more effective for the higher wavenumbers.

(2) The Kolmogorov scaling is overall the more effective of the two and this suggests that $(\epsilon^{\frac{1}{3}}k^{\frac{2}{3}})^{-1}$ is the dominant timescale in these two-time correlations.

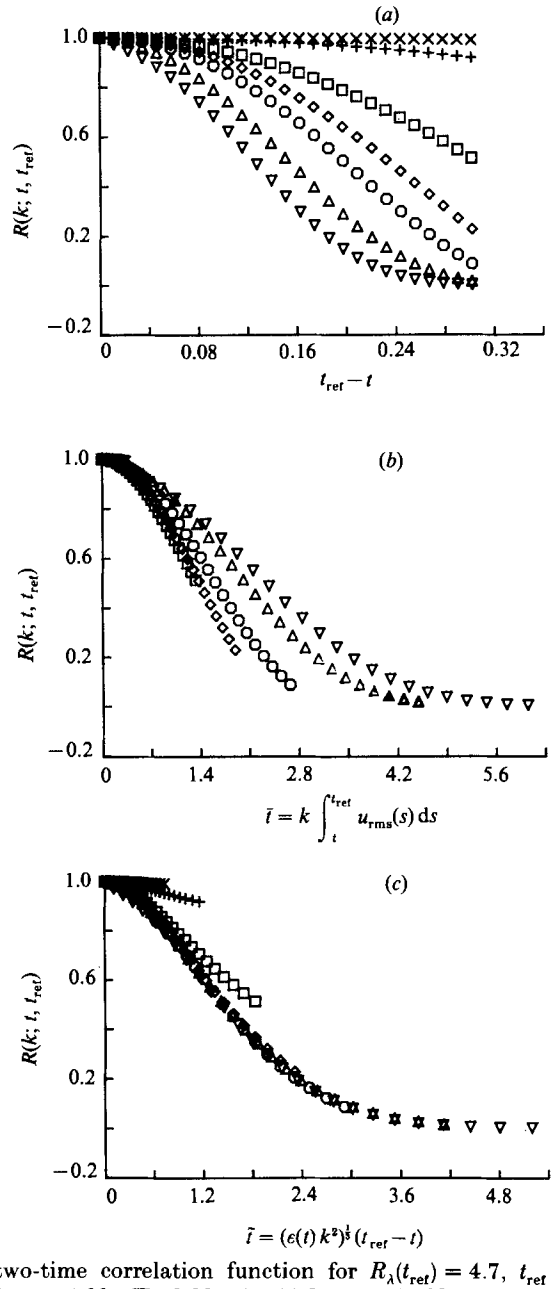


FIGURE 6. DIA two-time correlation function for $R_\lambda(t_{ref}) = 4.7$, $t_{ref} = 0.39$, for various wavenumbers: ×, 2.00; +, 4.00; □, 8.00; ◇, 11.31; ○, 16.00; △, 26.91; ▽, 38.05. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

(3) The discrepancy between the two scaling methods increases with increasing $R_\lambda(t_f)$, particularly at shorter lag times.

It is particularly interesting to compare figures 4 and 5, which are taken from the same run at different times. In figure 4, the calculation is not fully evolved, with $t_{ref} = 0.1$; and, although the Reynolds number is very large, convective scaling is (if anything) more effective than inertial scaling. In figure 5, the calculation is fully evolved with $t_{ref} = 0.6$, and the superiority of inertial scaling is apparent.

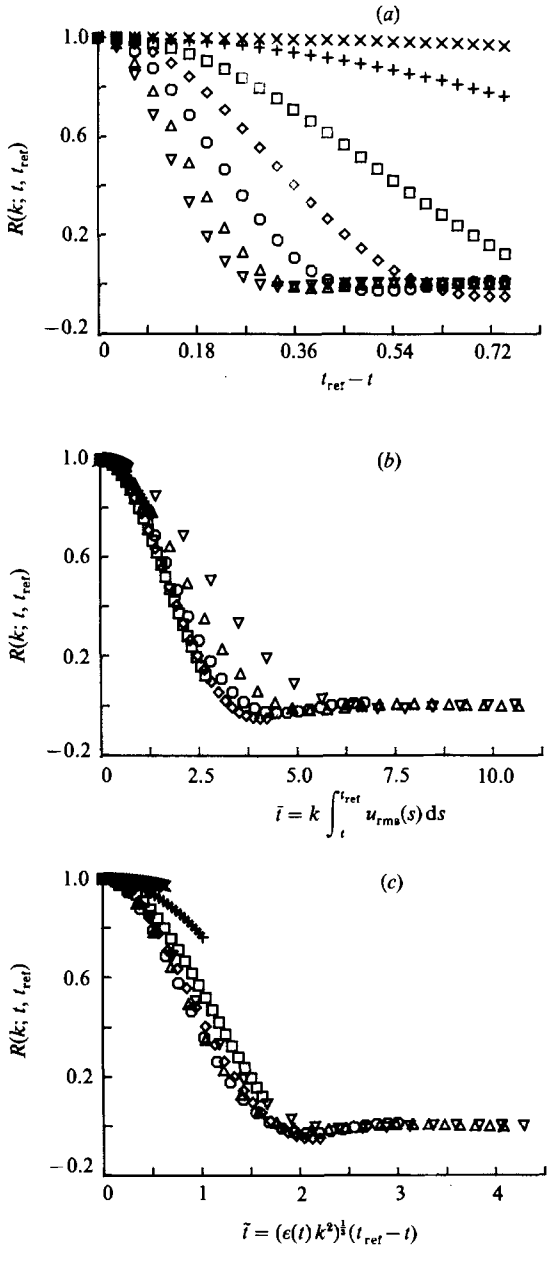


FIGURE 7. DIA two-time correltion function for $R_\lambda(t_{ref}) = 40.5$, $t_{ref} = 0.99$, for various wavenumbers: ×, 1.00; +, 2.00; □, 4.00; ◇, 6.35; ○, 10.08; △, 16.00; ▽, 25.40. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

Experimental results for two-time correlations are thin on the ground and no decisive comparison with the present results is possible. The general position has been summarized by Comte-Bellot & Corrsin (1971), who measured narrow-band Eulerian two-time correlations in grid turbulence and concluded that timescalings of the type given in (3.2) and (3.3) produced only a partial collapse of the $R(k;t,t')$ curves. Evidently, this provides some support for our own conclusions, although, of

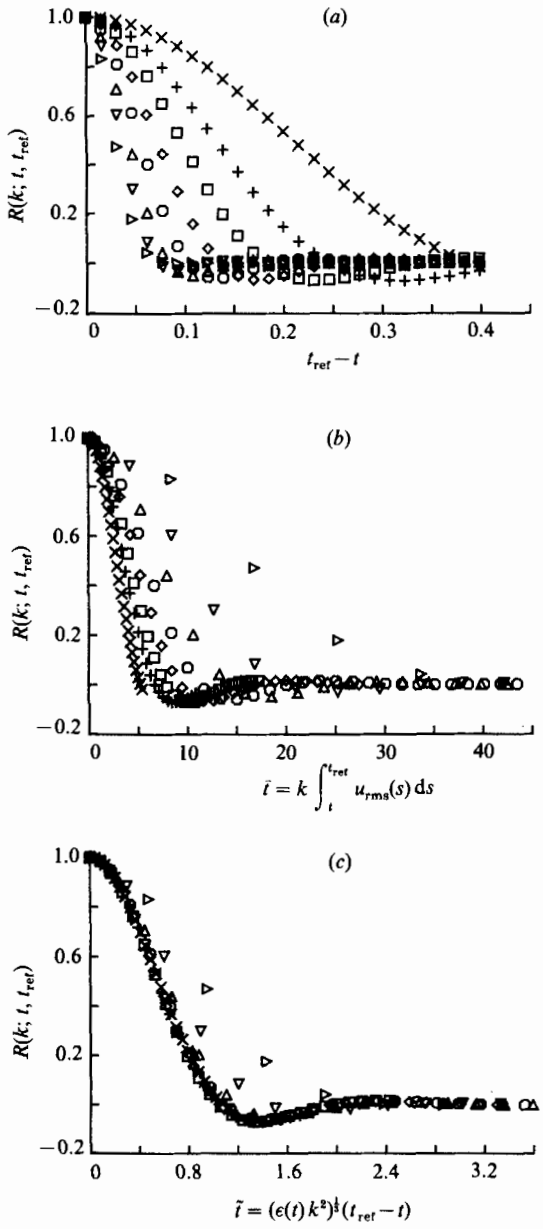


FIGURE 8. DIA two-time correlation function for $R_\lambda(t_{ref}) = 1039$, $t_{ref} = 0.6$, for various wavenumbers: ×, 2.28; +, 4.56; □, 7.24; ◇, 11.49; ○, 18.24; △, 28.96; ▽, 45.97; ▷, 91.95. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

course, we go further and argue that the Kolmogorov scaling becomes dominant over the convective-sweeping effect, as the Reynolds number increases.

However, our main motivation in studying two-time correlations has been to shed some light on the differences between LET and DIA (Kraichnan 1959); and we shall discuss this aspect as a separate issue in §4.

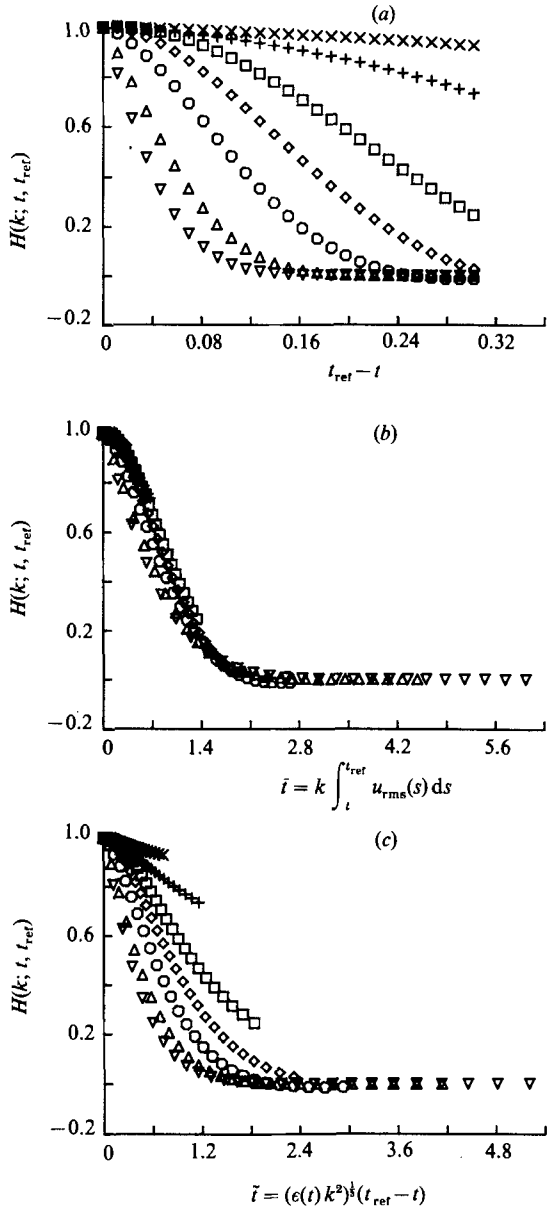


FIGURE 9. LET propagator function for $R_\lambda(t_{ref}) = 4.7$, $t_{ref} = 0.39$, for various wavenumbers; ×, 2.00; +, 4.00; □, 8.00; ◇, 11.31; ○, 16.00; △, 26.91; ▽, 38.05. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

### 3.3. *Propagator (or response) functions*

The picture presented by the results for the propagator function (or, in the case of DIA, the response function) was found to be quite similar to that for the two-time correlation, although there was one interesting difference. Using the same convention as for the correlation functions, we show some representative results in figures 9–11.

We found, for the LET propagator $H(k; t, t_{ref})$, that the convective scaling was the more effective at $R_\lambda = 4.7$; that there was little to choose between either scaling at $R_\lambda = 38.6$; and by $R_\lambda = 1009$ the Kolmogorov scaling has become the more effective.

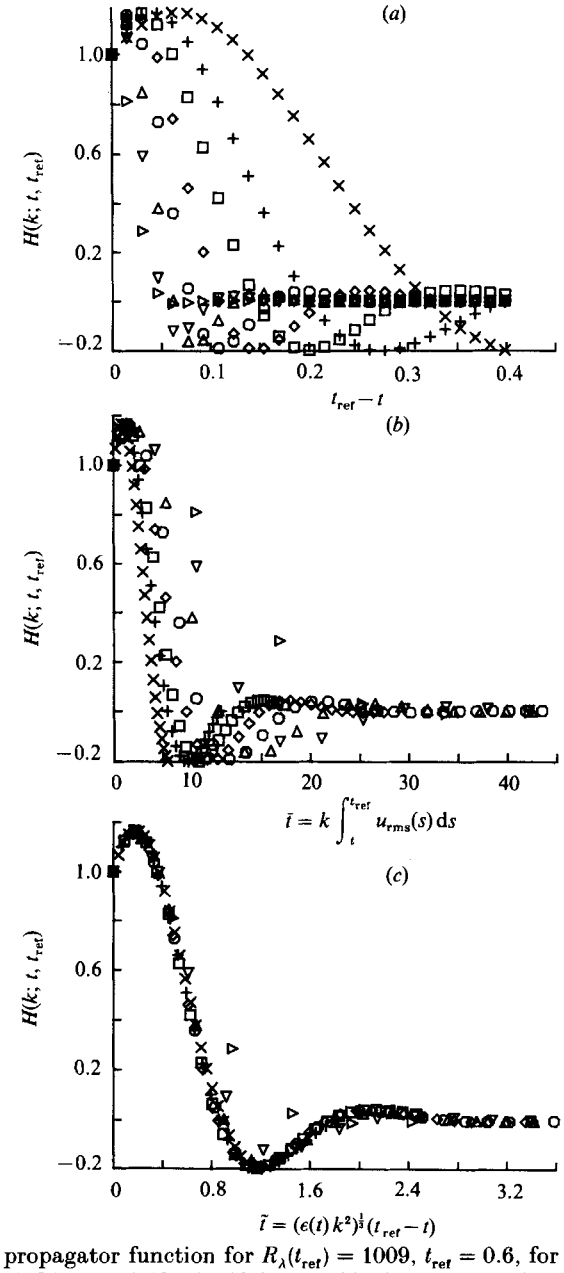


FIGURE 10. LET propagator function for $R_\lambda(t_{ref}) = 1009$, $t_{ref} = 0.6$, for various wavenumbers: ×, 2.28; +, 4.56; □, 7.24; ◇, 11.49; ○, 18.24; △, 28.96; ▽, 45.97; ▷, 91.95. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

In other words, at very low Reynolds numbers $(ku_{rms})^{-1}$ is the effective timescale for the propagator (the reverse being the case for the correlation function); while at large Reynolds numbers the propagator is (now like the correlation function) dominated by the $(\epsilon^{\frac{1}{3}}k^{\frac{2}{3}})^{-1}$ timescale. This behaviour is clearly illustrated by figures 9 and 10.

The DIA response function behaved in much the same way as the LET propagator. In figure 11, we show the DIA response function for $R_\lambda = 1039$, and the superiority of the inertial range scaling is really quite marked.

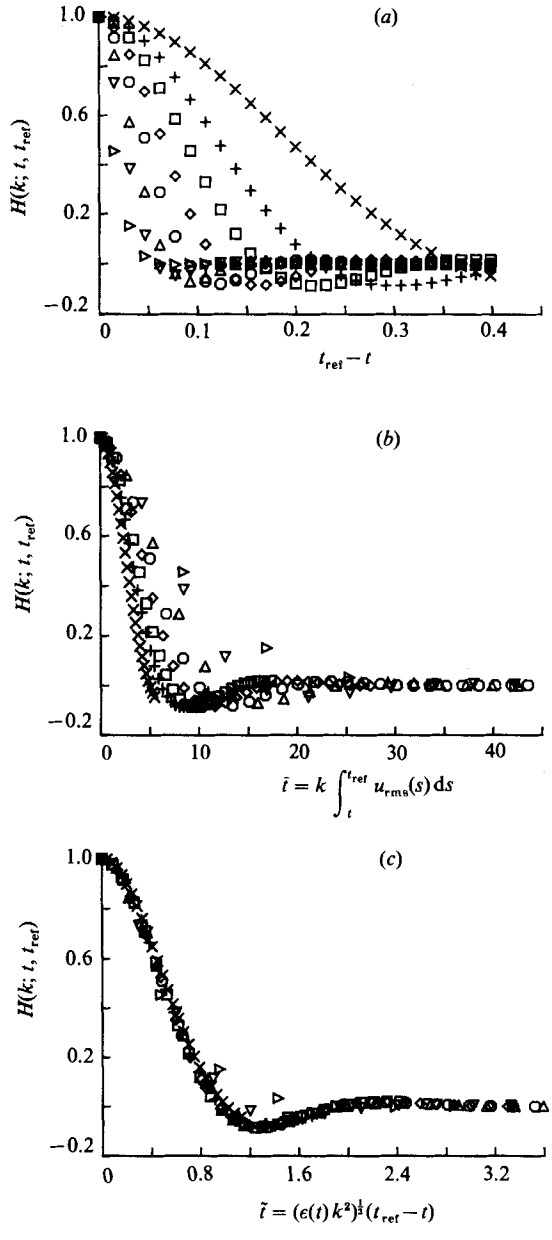


FIGURE 11. DIA response function for $R_\lambda(t_{ref}) = 1039$, $t_{ref} = 0.6$, for various wavenumbers: ×, 2.28; +, 4.56; □, 7.24; ◇, 11.49; ○, 18.24; △, 28.96; ▽, 45.97; ▷, 91.95. (*a*) Unscaled results, (*b*) convective scaling, (*c*) inertial scaling.

## 4. Two-time correlation functions as predicted by Eulerian DIA

We have seen that the collapse of results for various wavenumbers to a single curve is much more satisfactory in figures 6(*c*)–8(*c*) than in figures 6(*b*)–8(*b*). Thus, for DIA, as for LET, we conclude that the Kolmogorov scaling is more dominant than the convective form; and that this dominance appears to increase with the Reynolds number.

These results are surprising and may well prove controversial. Kraichnan (1959)

has argued that, for wavenumbers that are large compared to the energy-containing range, the Navier–Stokes equation may be linearized, leading to the approximate result that the correlation and response functions are both equal to the characteristic function associated with the single-point, single-time velocity distribution of the turbulence. A further assumption, that this distribution may be taken as Gaussian, results in the correlation and response functions each having the asymptotic form $\exp(-\frac{1}{2}v_0^2 k^2\tau^2)$, where $v_0$ is the r.m.s. velocity in any direction and $\tau = t - t'$. A similar analysis of the DIA (Kraichnan 1959) leads to the asymptotic form $J_1(2v_0 k\tau)/v_0 k\tau$. Thus, although there are some differences of detail between these two asymptotic forms, nevertheless, in both cases the characteristic time is $(v_0 k)^{-1}$.

Inevitably, our present results, if accepted, must raise some questions about the significance and range of validity of the above asymptotic forms. Furthermore, they must also impinge on some related – and widely held – beliefs in turbulence theory. These are the propositions that: (1) a turbulence theory should exhibit invariance under *random* Galilean transformations; and (2) moment formulations in Eulerian coordinates are intrinsically unsuited to the achievement of this result. They arose out of an explanation (Kraichnan 1964) of why Eulerian DIA gives a $k^{-\frac{3}{2}}$ spectrum, rather than the Kolmogorov $k^{-\frac{5}{3}}$ form, and they have been widely accepted (e.g. Orszag 1970; Leslie 1973).

In essence, the difficulty with DIA was explained in terms of a difference between single-time correlations and two-time correlations. The evolution of single-time correlations $R(k;t,t)$ (and hence the energy spectrum) should, it is argued, be governed by timescales of order $(\epsilon^{\frac{1}{3}}k^{\frac{2}{3}})^{-1}$. However, the DIA equations for $R(k;t,t)$ contain integrals over two-time correlation and response functions and (as we have said above) these are supposed to have characteristic times of order $(v_0 k)^{-1}$; hence (so the argument goes), the introduction of the spurious timescale $(v_0 k)^{-1})$ as the dominant time-scale for the relaxation of triple-correlations which determine $R(k;t,t)$. Thus, for the stationary case, the asymptotic spectrum becomes $E(k) \sim (\epsilon v_0)^{\frac{1}{2}} k^{-\frac{3}{2}}$.

The idea that this problem is equivalent to a violation of a statistical form of Galilean invariance again depends on the difference between single-time and two-time correlations. Suppose that a homogeneous turbulent field undergoes a Galilean transformation to a new frame moving at constant velocity $\boldsymbol{c}$ with respect to its old frame. That is, $\boldsymbol{c}$ is constant in both space and time, so that the effect of the transformation on the fluctuating turbulent field $\boldsymbol{u}(\boldsymbol{k},t)$ is merely a phase change, thus

$$\boldsymbol{u}(\boldsymbol{k},t) \to \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{c})t}\boldsymbol{u}(\boldsymbol{k},t). \tag{4.1}$$

Further suppose that $\boldsymbol{c}$ varies randomly from one realization to another (we are slurring over what we mean by 'realization' here: we shall be more specific in §5) and that it is specified by Gaussian statistics with zero mean and r.m.s. value $\tilde{c}$. We now consider the mean effect of random Galilean transformations on correlations of the Fourier amplitudes $\boldsymbol{u}(\boldsymbol{k},t)$, assuming that $\boldsymbol{u}$ and $\boldsymbol{c}$ are statistically independent random variables.

First consider single-time correlations of (for example) $\boldsymbol{u}(\boldsymbol{k},t)$ and $\boldsymbol{u}(\boldsymbol{k}',t)$. For homogeneous fields, averaging over $\boldsymbol{u}$ means that the correlation vanishes unless $\boldsymbol{k} = -\boldsymbol{k}'$. Hence, the phase changes introduced in the two modes, according to (4.1), cancel; i.e.

$$\mathrm{e}^{\mathrm{i}(\boldsymbol{k}+\boldsymbol{k}')\cdot\boldsymbol{c}t} = 1 \tag{4.2}$$

for each individual uniform convection velocity $\boldsymbol{c}$. Thus, the single-time correlation is unaffected by random Galilean transformation. Indeed, for $t = t'$ the random and deterministic Galilean transformations are indistinguishable. Note that this is a general result and applies to simultaneous correlation of any number of velocities.

Then consider $t \neq t'$. We no longer have the cancellation shown in (4.2), and this term has to be treated statistically. With the assumptions and restrictions made above, it is easily shown that

$$\langle\langle \boldsymbol{u}(\boldsymbol{k},t)\,\boldsymbol{u}(\boldsymbol{k}',t')\rangle\rangle_c \to \exp\{-\tfrac{1}{2}\tilde{c}^2k^2(t-t')^2\}\,\langle \boldsymbol{u}(\boldsymbol{k},t)\,\boldsymbol{u}(\boldsymbol{k}',t')\rangle, \tag{4.3}$$

where $\langle\ \rangle_c$ denotes an average over $\boldsymbol{c}$, and $\langle\ \rangle$ denotes the normal ensemble average over the turbulent fluctuating field $\boldsymbol{u}(\boldsymbol{k},t)$.

Now the difficulty with Eulerian DIA can be expressed in the following way. Single-time correlations should be invariant under Galilean transformation. Thus, the equation for $R(k;t,t)$, which contains a triple moment evaluated at $t = t' = t''$, should itself be invariant under random Galilean transformations. But, in the DIA formulation, the triple moment is expressed in terms of two-time correlation and response functions, which transform according to (4.3). Hence, the DIA triple moment – as given by (2.5), with expressions of the form of (4.3) substituted for each $H$ and $Q$ factor – will violate the requirement of random Galilean invariance† and exhibit a spurious decay even on the time diagonal $t = t' = t''$.

These arguments are given in much more detail by Kraichnan (1964) and have subsequently been developed to provide a rationale for the reworking of DIA in Lagrangian coordinates. As is well known, the result has been the successful Lagrangian-history theories (Kraichnan 1965; Kraichnan & Herring 1978).

Clearly, the idea of invariance under (specified) random Galilean transformations has provided a very powerful and successful method of closing the Navier–Stokes hierarchy. So much so, that it is nowadays often cited in the literature as a fundamental requirement of any analytical turbulence theory, much like (say) conservation of energy or momentum, or even *deterministic* Galilean invariance.

We feel that this introduction of a new invariance requirement into mechanics requires some more critical attention than it has received hitherto. Clearly, from the point of view of the present paper, we must be somewhat sceptical about the reliance on the notion that two-time correlations are dominated by energy-range convective sweeping times $(v_0 k)^{-1}$. Our results do not support this view and indeed, as we have seen, favour a scaling based on the Kolmogorov inertial-range timescale $(\epsilon^{\frac{1}{3}}k^{\frac{2}{3}})^{-1}$. Evidently, this – if correct – must raise at least some questions about the accepted reasons for the failure of Eulerian DIA.

Furthermore, in formulating his approach to the Lagrangian method, Kraichnan (1964) restricted his attention to some very carefully specified models of the full turbulence problem. If the velocity field $\boldsymbol{u}(\boldsymbol{k},t)$ in (4.3) is the ordinary (unrestricted) turbulent) field, then the averaging process which allows us to separate the averages over the distributions of $\boldsymbol{c}$ and $\boldsymbol{u}$ must be a very special one. Consideration of a simple example‡ shows that an ensemble average calculated after random Galilean transformations cannot in general be ergodic. Given a choice between ergodicity and

† We shall put forward a contrary view in §5.

‡ Consider a fluid in uniform steady motion, with velocity $V$ everywhere. That is, $V$ is constant in both space and time. Now apply a particular random Galilean transformation $c$, such that $\langle c\rangle \neq 0$. Then the ensemble average $\langle V+c\rangle$ is given by

$$\langle V+c\rangle = V+\langle c\rangle.$$

But the time average, $\overline{(V+c)}$ is

$$\overline{(V+c)} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(V+c)\,\mathrm{d}t = V+c.$$

Hence $\overline{(V+c)} \neq \langle V+c\rangle$, and the system is not ergodic.

invariance under random Galilean transformation we believe it is much more appropriate to select ergodicity. Moreover, the conflict between the two principles seems to us to raise questions about the fundamental status of the latter.

We shall discuss these points further in the next section, but before doing so, we note that our results for the DIA two-time correlations raise an even more interesting possibility. What is the DIA prediction for the energy spectrum at very high Reynolds numbers? There would now seem to be some doubt about the basis for the well-known $k^{-\frac{3}{2}}$ law.

We computed one-dimensional spectra for both DIA and LET, at Reynolds numbers large enough to show a Kolmogorov-type inertial range (plots of spectra are given in I, and not repeated here). We found that, at this rather large Reynolds number ($R_\lambda = 533$), the two theories were virtually indistinguishable and that both gave the Kolmogorov spectrum, to within the accuracy of the numerical methods.

It is perhaps worth pointing out that this surprising result (for DIA, that is) is not a consequence of taking $k^{-\frac{5}{3}}$ as the initial spectrum. We also took $k^{-\frac{3}{2}}$ as a trial spectrum, and found that *both* theories evolved away from this, to give $k^{-\frac{5}{3}}$ as the evolved spectrum.

Unfortunately, there appear to be no published calculations of DIA for freely decaying turbulence at high Reynolds numbers. Kraichnan (1964) obtained a $k^{-\frac{3}{2}}$ spectrum at $R_\lambda \sim 820$, but this calculation used exponential approximation for $R(k;t,t')$ and $G(k;t,t')$, and employed random forcing at low wavenumbers in order to obtain a statistically steady state. It is, therefore, not immediately comparable to our own calculation (although, we should perhaps mention that we used the same numerical methods). Also, Kraichnan found that by restricting the convolution integrals over wavenumber in appropriate ways, he could force DIA to give a $k^{-\frac{5}{3}}$ spectrum.

We think that our result for DIA is perhaps just due to our value of Taylor–Reynolds number – large though it is at $R_\lambda \sim 1000$ – not being large enough for Kraichnan's asymptotic analysis to apply. This seems more likely than the possibility that we have a numerical artefact, in which integrals over $\boldsymbol{j}$ are evaluated with some lower bound which is simply proportional to the labelling wavenumber $\boldsymbol{k}$.

## 5. Random Galilean invariance (RGI)

In the previous section, we summarized Kraichnan's diagnosis of the failure of DIA to maintain the property of random Galilean invariance. In this section we shall examine the idea of RGI in a slightly more formal way. The specific question we shall have in mind is this: does a postulate of RGI add anything of a non-trivial nature to the usual deterministic requirement?

We shall begin by saying what we mean by random Galilean invariance! Let us refer to our usual reference frame as $S$. Then in $S$, we perform experiments – involving many realizations of the velocity field $\boldsymbol{u}(\boldsymbol{k}, t)$ – and, by ensemble averaging, obtain correlation and propagator functions: $Q(k;t,t')$ and $H(k;t,t')$.

These statistical quantities – along with all higher-order moments and the relationships between them – may be transformed to a new coordinate system, which is defined through a frame of reference $S_0$, moving with constant velocity $\boldsymbol{c}_0$ relative to $S$. It is a cardinal principle of physics that the description of any dynamical process should be the same in both systems. For example, the Navier–Stokes equation is form-invariant under such a transformation. This property is known as

'Galilean invariance' (GI) and is the classical form of Lorentz invariance, applicable in systems where all speeds are much smaller than the speed of light.

How then do we extend this concept to embrace the idea of *random* Galilean invariance? Let us suppose that we transform the results of our experiment in $S$ to many different frames of reference $S_0, S_1, S_2 \ldots$, moving with velocities $\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2 \ldots$, relative to $S$. The velocity of transformation $\boldsymbol{c}$ is taken to be a random variable with realizations $\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2, \ldots$, and so on. The set of such realizations $\{\boldsymbol{c}\}$ defines the ensemble, and the distribution $P(\boldsymbol{c})$ has to be specified arbitrarily.

The result is a static ensemble, in which each realization is its own time average. As we saw previously, such an ensemble is non-ergodic, and has no dynamical significance. But this need not concern us here, for we are interested in the RGI itself, rather than in its possible connection with energy transfer and other convective processes in turbulence.

In particular, the formation of ensemble averages – again denoted by $\langle\ \rangle_c$ – seems to present no problem. Any realization of the velocity field $\boldsymbol{u}(\boldsymbol{k}, t)$ in $S$ may be labelled by the random variable, $\boldsymbol{c}$, and averaged over $P(\boldsymbol{c})$. From (4.1) we have

$$\langle \boldsymbol{u}^{(c)}(\boldsymbol{k}, t)\rangle_c = \langle e^{i\boldsymbol{k}\cdot\boldsymbol{c}t}\rangle_c \, u(\boldsymbol{k}, t). \tag{5.1}$$

It follows at once from the definition of the two-time velocity covariance that we may write

$$\langle Q^{(c)}(\boldsymbol{k}; t, t')\rangle_c = \langle e^{i(\boldsymbol{k}\cdot\boldsymbol{c})(t-t')}\rangle_c \, Q(\boldsymbol{k}; t, t'). \tag{5.2}$$

We shall also assume that propagator or response functions behave in a similar way.

At this point we should note that the RGI of single-time correlations follows at once when we put $t = t'$ on both sides of (5.2) (also see (4.2)) but that the result contained in (4.3) requires the additional assumption that $P(\boldsymbol{c})$ is Gaussian, with zero mean and r.m.s. value $\tilde{c}$.

Even without specifying a form for $P(\boldsymbol{c})$, we can note that every random Galilean transformation is identical in form and hence one may state some properties of the random Galilean ensemble as follows:

(1) The Navier–Stokes equation is form invariant under every Galilean transformation which makes up the ensemble and is hence RGI.

(2) Simultaneous correlations of velocities at two or more space points (as obtained by averaging over the turbulent ensemble in $S$) are invariant under every Galilean transformation which makes up the set $\{\boldsymbol{c}\}$ and hence are RGI.

(3) Non-simultaneous correlations of velocities at two or more space points (as obtained by averaging over the turbulent ensemble in $S$) are not invariant under Galilean transformation. Accordingly, any further averaging over the ensemble $\{\boldsymbol{c}\}$ will tend to 'smear out' the turbulent correlation as in (5.2).

(4) On the other hand, relations between moments, as derived by averaging the Navier–Stokes equation over the turbulent ensemble in $S$, are form-invariant under every transformation making up the Galilean ensemble. Hence it follows that the equations relating moments of different order are RGI, even although the individual moments themselves are not.

Property (4) – like (2) – is an exact consequence of homogeneity and runs counter to the views of Kraichnan, as discussed in the previous section. Essentially, the general corollary of (4) is that any theory which fails to be RGI, also fails to be GI.

To be more specific, it can easily be shown that (2.3) for the correlation *function* is invariant under any Galilean transformation and hence under all such

transformations. This is true irrespective of whether we take the exact form of $P(k;t,t')$, in terms of the triple moment, or whether we take the second-order closure given by (2.5).

Thus, it follows that DIA (like all such Eulerian closures) is invariant under Galilean transformation. The further property of random Galilean invariance – as we have defined it here – follows in an entirely trivial fashion. It, therefore, seems to us that Kraichnan's definition of RGI must in some way be more special than ours and must involve some approximations. If we write (2.5) schematically and average over the ensemble of Galilean transformations, thus:

$$\langle P(k;t,t')\rangle_c \propto \langle H(k;t',s)\,Q(j;t,s)\,Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s)\rangle_c,$$

Kraichnan's arguments would seem to require the further step

$$\langle HQQ\rangle_c \sim \langle H(k;t',s)\rangle_c \langle Q(j;t,s)\rangle_c \langle Q(|\boldsymbol{k}-\boldsymbol{j}|;t,s)\rangle_c.$$

Then, even for $t = t'$, substitution from (4.3) for each of the bracketed terms would result in time dependences that would not cancel the phase factor on the left-hand side of (2.3).

## 6. Conclusion

The LET prediction of the skewness factor has been compared with other theories and with numerical simulations. The overall result may be seen as quite encouraging, although the disagreement between the various simulations needs to be resolved before there can be any definitive test of analytical theories. Evidently, there is a need for more numerical simulations to be carried out, as such 'computer experiments' hold out the possibility of better and more accurate tests of analytical theories than have been possible hitherto. The unconvinced reader need only refer to the compilation of data from laboratory and environmental experiments, by Tavoularis *et al.* (1978), to see the force of this observation.

We may also hope that future numerical simulations will provide useful information on the two-time correlation and response functions. At present, the only published results for $R(k;t,t')$ appear in the pioneering simulation of Orszag & Patterson (1972). But, although we may note in passing that the LET predictions agree quite well with these results, they are, in fact, too limited in both Reynolds number and wavenumber to shed any light on some of the questions raised in the preceding sections.

Turning, lastly, to the more controversial parts of this work, we note that our result of a $k^{-\frac{5}{3}}$ spectrum for DIA is not as surprising as it might first appear. It has long been known that a closure approximation for the correlation function, of the form given by (2.5), is rigorously consistent with the Kolmogorov distribution. It was shown by Edwards (1965), for the stationary case, that at infinite Reynolds numbers the kernel on the right-hand side of (2.5) gives delta functions which balance the input at $k = 0$ and the viscous dissipation at $k = \infty$. The problem – for DIA – arises with the response equation, in which the integration over $\boldsymbol{j}$ diverges at the origin. This is the well-known 'infra-red divergence' of turbulence theory and is discussed by Leslie (1973, p. 103). This analysis certainly suggests that, at finite Reynolds numbers, numerical computation of either the Edwards theory or Kraichnan's DIA will lead to a $k^{-\frac{5}{3}}$ inertial range; but with a value for the Kolmogorov constant which increases without limit as the Reynolds number is allowed to tend to infinity.

Intuitively, it may seem that a divergence at $k = 0$ must have some connection with a lack of Galilean invariance. But, at the same time, it must be borne in mind that all these theories *are* Galilean invariant, as an exact consequence of homogeneity. In addition, if one accepts our present results that two-time correlations are *not* dominated by convective effects of large eddies, then what can we put in place of Kraichnan's (1964) analysis?

An alternative has been around for quite some time. The picture due to Edwards (1965) was that the correlation equation was compatible with the Kolmogorov distribution, because terms which were individually divergent cancelled each other at the singular points, thus giving a well-behaved integral. No such cancellation occurred in the response equation. Later, McComb (1974, 1976) showed that this could be rectified by noting that the entire nonlinear term behaved as both input and output, according to the particular circumstances. The result was a response equation which had the necessary cancellations and hence was well behaved.

Although couched in terms of the Edwards theory, all the physical arguments used (McComb 1974, 1976) are equally applicable to DIA.† We are, therefore, tempted to speculate that since (1) the basic ansatz of the DIA is the relationship of the velocity field to the stirring forces, and (2) the renormalized version of this relationship is equivalent to the original response function of the Edwards (1964) theory, the principal defect of DIA is a failure to renormalize the stirring forces. This would be in line with the work of Forster, Nelson & Stephen (1977), who applied renormalization group ideas to the Navier–Stokes equation. Although their work is of limited applicability, nonetheless they showed that a full renormalization program for perturbation theory required the renormalization of three quantities: the interaction strength, the viscosity, and the stirring forces. Only the first two are taken into account in DIA.

We would like to thank David Leslie for reading an earlier version of this paper and making some helpful comments. We are particularly grateful for his raising the question of the behaviour of the DIA energy spectrum at large Reynolds numbers. We would also like to thank R. H. Kraichnan and J. R. Herring for a helpful correspondence, and Alex Watt for some stimulating discussions. We also wish to thank the Science and Engineering Research Council for financial support.

## REFERENCES

BATCHELOR, G. K. 1971 *The Theory of Homogeneous Turbulence* (2nd edn). Cambridge University Press.

CLARK, R. A., FERZIGER, J. H. & REYNOLDS, W. C. 1979 Evaluation of subgrid scale models using an accurately simulated turbulent flow. *J. Fluid Mech.* **91**, 1.

COMTE-BELLOT, G. & CORRSIN, S. 1971 Simple Eulerian time correlation of full- and narrow-band velocity signals in grid generated 'isotropic' turbulence. *J. Fluid Mech.* **48**, 273.

EDWARDS, S. F. 1964 The statistical dynamics of homogeneous turbulence. *J. Fluid Mech.* **18**, 239.

EDWARDS, S. F. 1965 Turbulence in hydrodynamics and plasma physics. *Intl Conf. Plasma Phys., Trieste*, p. 595. IAEA, Vienna.

FORSTER, D., NELSON, D. R. & STEPHEN, M. J. 1977 Long-distance and long-time properties of a randomly stirred fluid. *Phys. Rev.* **A16**, 732.

HERRING, J. R. & KERR, R. M. 1982 Comparison of direct numerical simulations with predictions of two-point closures for isotropic turbulence convecting a passive scalar. *J. Fluid Mech.* **118**, 205.

† Apparently this was recognized – at least, implicitly – by Kraichnan (1976) in defining the subgrid eddy viscosity for large-eddy simulation of isotropic turbulence.

HERRING, J. R. & KRAICHNAN, R. H. 1979 A numerical comparison of velocity-based and strain-based Lagrangian-history turbulence approximations. *J. Fluid Mech.* **91**, 581.

KERR, R. M. 1985 Higher-order derivative correlations and the alignment of small-scale structures in isotropic numerical turbulence. *J. Fluid Mech.* **153**, 31.

KRAICHNAN, R. H. 1959 The structure of isotropic turbulence at very high Reynolds numbers. *J. Fluid Mech.* **5**, 497.

KRAICHNAN, R. H. 1964 Kolmogorov's hypotheses and Eulerian turbulence theory. *Phys. Fluids* **7**, 1723.

KRAICHNAN, R. H. 1965 Lagrangian-history closure approximation for turbulence. *Phys. Fluids* **8**, 575.

KRAICHNAN, R. H. 1976 Eddy viscosity in two- and three-dimensions. *J. Atmos. Sci.* **33**, 1521.

KRAICHNAN, R. H. & HERRING, J. R. 1978 A strain-based Lagrangian-History turbulence theory. *J. Fluid Mech.* **88**, 355.

LESLIE, D. C. 1973 *Developments in the Theory of Turbulence.* Oxford University Press.

MCCOMB, W. D. 1974 A local energy-transfer theory of isotropic turbulence. *J. Phys.* A: *Math. Nucl. Gen.* **7**, 632.

MCCOMB, W. D. 1976 The inertial-range spectrum from a local energy-transfer theory of isotropic turbulence. *J. Phys.* A: *Math. Gen.* **9**, 179

MCCOMB, W. D. 1978 A theory of time-dependent, isotropic turbulence. *J. Phys.* A: *Math. Gen.* **11**, 613.

MCCOMB, W. D. & SHANMUGASUNDARAM, V. 1984 Numerical calculation of decaying isotropic turbulence using the LET theory. *J. Fluid Mech.* **143**, 95 (referred to as I).

ORSZAG, S. A. 1970 Analytical theories of turbulence. *J. Fluid Mech.* **41**, 363.

ORSZAG, S. A. & PATTERSON, G. S. 1972 Numerical simulation of three-dimensional homogeneous isotropic turbulence. *Phys. Rev. Lett.* **28**, 76.

SIGGIA, E. D. & PATTERSON, G. S. 1978 Intermittency effects in a numerical simulation of stationary three-dimensional turbulence. *J. Fluid Mech.* **86**, 567.

TAVOULARIS, S., BENNETT, J. C. & CORRSIN, S. 1978 Velocity-derivative skewness in small Reynolds number, nearly isotropic turbulence. *J. Fluid Mech.* **88**, 63.